

On Generalized Douglas-Weyl (α, β) -Metrics*

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Abstract

In this paper, we study generalized Douglas-Weyl (α, β) -metrics. Suppose that an regular (α, β) -metric F is not of Randers type. We prove that F is a generalized Douglas-Weyl metric with vanishing S-curvature if and only if it is a Berwald metric. Moreover by ignoring the regularity, if F is not a Berwald metric then we find a family of almost regular Finsler metrics which is not Douglas nor Weyl. As its application, we show that generalized Douglas-Weyl square metric or Matsumoto metric with isotropic mean Berwald curvature are Berwald metrics.

Keywords: Generalized Douglas-Weyl metric, Weyl metric, Douglas metric, S -curvature.¹

1 Introduction

In [4], Cheng proved that every non-Randers type (α, β) -metric of scalar flag curvature with vanishing S-curvature is a Berwald metric. Finsler metrics of scalar flag curvature are called Weyl metrics. In [12], Sakaguchi showed that every Weyl metric is a generalized Douglas-Weyl metric. This motivates us to study generalized Douglas-Weyl (α, β) -metric with vanishing S-curvature.

For considering generalized Douglas-Weyl metric, let us introduce some Finslerian notions. Let (M, F) be a Finsler manifold. In local coordinates, a curve $c(t)$ is a geodesic if and only if its coordinates $(c^i(t))$ satisfy $\ddot{c}^i + 2G^i(\dot{c}) = 0$, where the local functions $G^i = G^i(x, y)$ are called the spray coefficients. F is called a Berwald metric, if G^i are quadratic in $y \in T_x M$ for any $x \in M$ or equivalently $2G^i = \Gamma_{jk}^i(x)y^j y^k$. As a generalization of Berwald curvature, Bácsó-Matsumoto introduced the notion of Douglas metrics which are projective invariants in Finsler geometry [2]. F is called a Douglas metric if $G^i = \frac{1}{2}\Gamma_{jk}^i(x)y^j y^k + P(x, y)y^i$.

Other than Douglas metrics, there is another projective invariant in Finsler geometry, namely

$$D^i_{jkl|m}y^m = T_{jkl}y^i$$

that is hold for some tensor T_{jkl} , where $D^i_{jkl|m}$ denotes the horizontal covariant derivatives of Douglas curvature D^i_{jkl} with respect to the Berwald connection of F . For a manifold M , let $\mathcal{GDW}(M)$ denotes the class of all Finsler metrics satisfying in above relation for some tensor T_{jkl} . In [3], Bácsó-Papp showed that $\mathcal{GDW}(M)$ is closed under projective changes. Then, Najafi-Shen-Tayebi characterized generalized Douglas-Weyl Randers metrics [10]. Recently, the authors with Peyghan prove that all generalized Douglas-Weyl spaces with vanishing Landsberg curvature have vanishing the quantity \mathbf{H} [19].

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In order to find explicit examples of generalized Douglas-Weyl metrics, we consider (α, β) -metrics. This class of metrics was first introduced by Matsumoto which appear iteratively in formulating Physics, Mechanics, Biology and Ecology, etc [1][8]. An (α, β) -metric is a Finsler metric of the form $F := \alpha\phi(s)$, $s = \beta/\alpha$, where $\phi = \phi(s)$ is a C^∞ on $(-b_0, b_0)$, $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M . In particular, $\phi = c_1\sqrt{1+c_2s^2} + c_3s$, $s = \beta/\alpha$, are called Randers type metrics where $c_1 > 0$, c_2 and c_3 are constant.

The notion of S-curvature is originally introduced by Shen for the volume comparison theorem which interacts with other Riemannian and non-Riemannian curvatures [13][17][18]. The Finsler metric F is said to be of isotropic S-curvature if $\mathbf{S} = (n+1)cF$, where $c = c(x)$ is a scalar function on M . If c is a constant, then F is said to be of constant S-curvature. The studies how that the S-curvature plays a very important role in Finsler geometry [9]. Recently, Shen proved that every negatively curved Finsler metric with constant S-curvature on a closed manifold must be Riemannian [15]. In [18], it is proved that every isotropic Berwald metric has isotropic S-curvature. The Finsler metrics with vanishing S-curvature are of some important geometric structures which deserve to be studied deeply. For example, Shen proved that the Bishop-Gromov volume comparison holds for Finsler manifolds with vanishing S-curvature [14].

In this paper, we characterize the generalized Douglas-Weyl (α, β) -metrics with vanishing S-curvature. More precisely, we prove the following.

Theorem 1.1. *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be a regular (α, β) -metric on a manifold M . Suppose that F is not a Finsler metric of Randers type. Then F is a generalized Douglas-Weyl metric with vanishing S-curvature if and only if it is a Berwald metric.*

According to Cheng's theorem (Theorem 4 in [4]), every non-Randers type (α, β) -metric on a manifold M of dimension $n \geq 3$ is Weyl metric with vanishing S-curvature if and only if it is a Berwald metric with vanishing flag curvature. Here, we weaken Cheng's condition on the Weyl metrics to the generalized Douglas-Weyl metrics. We also delete the dimension's condition on manifold. Then Theorem 1.1, is an extension of Cheng's theorem.

We must mention that Theorem 1.1 does not hold for Finsler metrics of Randers type. The family of Randers metrics on S^3 constructed by Bao-Shen are generalized Douglas-Weyl metric with $\mathbf{S} = 0$ which are not Berwaldian (see [14], page 164).

In Theorem 1.1, vanishing of S-curvature is necessary. For example, consider following Finsler metric on the unit ball \mathbb{B}^n

$$F := \frac{(\sqrt{(1-|x|^2)}|y|^2 + \langle x, y \rangle^2 + \langle x, y \rangle)^2}{(1-|x|^2)^2 \sqrt{(1-|x|^2)}|y|^2 + \langle x, y \rangle^2},$$

where $|\cdot|$ and $\langle \cdot, \cdot \rangle$ denote the Euclidean norm and inner product in \mathbb{R}^n , respectively. This metric is projectively flat with flag curvature $\mathbf{K} = 0$ (see page 96 in [1]). Therefore F is a Weyl metric and then it is a generalized Douglas-Weyl metric, also. F satisfies $\mathbf{S} \neq 0$ and is not a Berwald metric.

Theorem 1.1, may not be hold for an (α, β) -metric of constant S-curvature. For example, at a point $x = (x^1, x^2) \in R^2$ and in the direction $y = (y^1, y^2) \in T_x R^2$, consider Riemannian metric $\alpha(x, y) = \sqrt{(y^1)^2 + e^{2x^1}(y^2)^2}$ and one form $\beta(x, y) := y^1$. Then $s_{ij} = 0$, $r_{ij} = a_{ij} - b_i b_j$ and $\epsilon = b = 1$. Thus if $\phi = \phi(s)$ satisfies (3) for some constant k , then $F = \alpha\phi(\beta/\alpha)$ has constant S-curvature $\mathbf{S} = 3kF$. Since every two-dimensional metric is a Weyl metric, then F is a generalized Douglas-Weyl metric while it is not a Berwald metric.

Let $\phi = \phi(s)$ satisfy $\phi(s) > 0$ and $\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0$, where $(|s| \leq b \leq b_0)$. A function $F = \alpha\phi(s)$ is called an almost regular (α, β) -metric if β satisfies that $\|\beta_x\|_\alpha \leq b_0$, $\forall x \in M$ [16]. An almost regular (α, β) -metric $F = \alpha\phi(s)$ might be singular (even not

defined) in the two extremal directions $y \in T_x M$ with $\beta(x, y) = \pm b_0 \alpha(x, y)$. By assumptions of Theorem 1.1, we find that if F is almost regular and not Berwaldian then it reduces to

$$\phi = c \exp \left[\int_0^s \frac{kt + q\sqrt{b^2 - t^2}}{1 + kt^2 + qt\sqrt{b^2 - t^2}} dt \right],$$

where $c > 0$, $q > 0$ and k are real constants. The above Finsler metric is not a Douglas metric nor Weyl metric. It is not Landsberg metric, also (see Remark 4.1).

Taking twice vertical covariant derivatives of S -curvature \mathbf{S} gives rise E -curvature \mathbf{E} . The Finsler metric F is said to have isotropic mean Berwald curvature if $\mathbf{E} = \frac{n+1}{2} c F \mathbf{h}$, where $c = c(x)$ is a scalar function on M and $\mathbf{h} = h_{ij} dx^i \otimes dx^j$ is the angular metric. Among the (α, β) -metrics, the square metric $F = \alpha + 2\beta + \beta^2/\alpha$ and the Matsumoto metric $F = \alpha^2/(\alpha - \beta)$ are significant metric which constitute a majority of actual research.

Corollary 1.1. *Let F be a generalized Douglas-Weyl square metric or Matsumoto metric. Suppose that F has isotropic mean Berwald curvature. Then it reduces to a Berwald metric.*

In [5], Cheng-Lu proved that every weakly Berwald square metric or Matsumoto metric of scalar flag curvature is a Berwald metric. Then Corollary 1.1 is an extension of Cheng-Lu's theorem.

2 Preliminary

Given a Finsler manifold (M, F) , then a global vector field \mathbf{G} is induced by F on $TM_0 := TM - \{0\}$, which in a standard coordinate (x^i, y^i) for TM_0 is given by $\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y) \frac{\partial}{\partial y^i}$, where

$$G^i := \frac{1}{4} g^{il} \left[\frac{\partial^2 (F^2)}{\partial x^k \partial y^l} y^k - \frac{\partial (F^2)}{\partial x^l} \right], \quad y \in T_x M.$$

The \mathbf{G} is called the spray associated to F . For $y \in T_x M_0$, define $\mathbf{B}_y : T_x M \otimes T_x M \otimes T_x M \rightarrow T_x M$, $\mathbf{E}_y : T_x M \otimes T_x M \rightarrow \mathbb{R}$ and $\mathbf{D}_y : T_x M \otimes T_x M \otimes T_x M \rightarrow T_x M$ by $\mathbf{B}_y(u, v, w) := B^i_{jkl}(y) u^j v^k w^l \frac{\partial}{\partial x^i} |_{x}$, $\mathbf{E}_y(u, v) := E_{jk}(y) u^j v^k$ and $\mathbf{D}_y(u, v, w) := D^i_{jkl}(y) u^j v^k w^l \frac{\partial}{\partial x^i} |_{x}$, where

$$B^i_{jkl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}, \quad E_{jk} := \frac{1}{2} B^m_{jkm}, \quad D^i_{jkl} := B^i_{jkl} - \frac{\partial^3}{\partial y^j \partial y^k \partial y^l} \left[\frac{2}{n+1} \frac{\partial G^m}{\partial y^m} y^i \right].$$

The \mathbf{B} , \mathbf{E} and \mathbf{D} are called the Berwald curvature, mean Berwald curvature and Douglas curvature, respectively. Then F is called a Berwald metric, weakly Berwald metric and Douglas metric if $\mathbf{B} = \mathbf{0}$, $\mathbf{E} = \mathbf{0}$ and $\mathbf{D} = \mathbf{0}$, respectively. If $\mathbf{E} = \mathbf{0}$, then $\mathbf{D} = \mathbf{B}$.

The notion of Riemann curvature for Riemann metrics can be extended to Finsler metrics. For a non-zero vector $y \in T_x M_0$, the Riemann curvature is a family of linear transformation $\mathbf{R}_y : T_x M \rightarrow T_x M$ with homogeneity $\mathbf{R}_{\lambda y} = \lambda^2 \mathbf{R}_y$, $\forall \lambda > 0$ which is defined by $\mathbf{R}_y(u) := R^i_k(y) u^k \frac{\partial}{\partial x^i}$, where

$$R^i_k(y) = 2 \frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k} y^j + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j} \frac{\partial G^j}{\partial y^k}.$$

The family $\mathbf{R} := \{\mathbf{R}_y\}_{y \in TM_0}$ is called the Riemann curvature. A Finsler metric F is said to be R-quadratic if \mathbf{R}_y is quadratic in $y \in T_x M$ at each point $x \in M$.

Let $F := \alpha\phi(s)$, $s = \beta/\alpha$, be an (α, β) -metric on a manifold M , where $\phi = \phi(s)$ is a C^∞ function on the $(-b_0, b_0)$ with certain regularity, $\alpha = \sqrt{a_{ij}y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M . Define $b_{i|j}$ by

$$b_{i|j}\theta^j := db_i - b_j\theta_i^j,$$

where $\theta^i := dx^i$ and $\theta_i^j := \Gamma_{ik}^j dx^k$ denote the Levi-Civita connection form of α . Let

$$r_{ij} := \frac{1}{2}(b_{i|j} + b_{j|i}), \quad s_{ij} := \frac{1}{2}(b_{i|j} - b_{j|i}).$$

Clearly, β is closed if and only if $s_{ij} = 0$. Put

$$\begin{aligned} r_{i0} &:= r_{ij}y^j, \quad r_{00} := r_{ij}y^i y^j, \quad r_j := b^i r_{ij}, \quad r_0 := r_j y^j, \\ s_{i0} &:= s_{ij}y^j, \quad s_j := b^i s_{ij}, \quad s_0 := s_j y^j. \end{aligned}$$

Now, let $G^i = G^i(x, y)$ and $\bar{G}_\alpha^i = \bar{G}_\alpha^i(x, y)$ denote the coefficients of F and α respectively in the same coordinate system. By definition, we have

$$G^i = G_\alpha^i + \alpha Q s_0^i + (-2Q\alpha s_0 + r_{00})(\Theta \frac{y^i}{\alpha} + \Psi b^i), \quad (1)$$

where

$$Q := \frac{\phi'}{\phi - s\phi'}, \quad \Theta := \frac{\phi\phi' - s(\phi\phi'' + \phi'\phi')}{2\phi[(\phi - s\phi') + (b^2 - s^2)\phi'']}, \quad \Psi := \frac{1}{2} \frac{\phi''}{(\phi - s\phi') + (b^2 - s^2)\phi''}.$$

For a Finsler metric F on an n -dimensional manifold M , the Busemann-Hausdorff volume form $dV_F = \sigma_F(x)dx^1 \cdots dx^n$ is defined by

$$\sigma_F(x) := \frac{\text{Vol}(\mathbb{B}^n(1))}{\text{Vol}\left[(y^i) \in R^n \mid F\left(y^i \frac{\partial}{\partial x^i} \Big|_x\right) < 1\right]}.$$

Let G^i denote the geodesic coefficients of F in the same local coordinate system. The S-curvature is defined by

$$\mathbf{S}(\mathbf{y}) := \frac{\partial G^i}{\partial y^i}(x, y) - y^i \frac{\partial}{\partial x^i} \left[\ln \sigma_F(x) \right],$$

where $\mathbf{y} = y^i \frac{\partial}{\partial x^i} \Big|_x \in T_x M$. \mathbf{S} said to be isotropic if there is a scalar functions $c = c(x)$ on M such that $\mathbf{S} = (n+1)cF$.

Now, let $\phi = \phi(s)$ be a positive C^∞ function on $(-b_0, b_0)$. For a number $b \in [0, b_0)$, put

$$\Phi := -(Q - sQ')[n\Delta + 1 + sQ] - (b^2 - s^2)(1 + sQ)Q''$$

where $\Delta := 1 + sQ + (b^2 - s^2)Q'$. In [6], Cheng-Shen characterize (α, β) -metrics with isotropic S-curvature.

Lemma 2.1. ([6]) Let $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$, be an (α, β) -metric on a n -dimensional manifold M . Suppose that $\phi \neq c_1\sqrt{1 + c_2 s^2} + c_3 s$ for any constant $c_1 > 0$, c_2 and c_3 . Then F is of isotropic S-curvature if and only if one of the following holds

(a) β satisfies

$$r_{ij} = \varepsilon(b^2 a_{ij} - b_i b_j), \quad s_j = 0, \quad (2)$$

where $\varepsilon = \varepsilon(x)$ is a scalar function, $b := \|\beta_x\|_\alpha$ and $\phi = \phi(s)$ satisfies

$$\Phi = -2(n+1)k \frac{\phi \Delta^2}{b^2 - s^2}, \quad (3)$$

where k is a constant. In this case, $\mathbf{S} = (n+1)cF$ with $c = k\varepsilon$.
(b) β satisfies

$$r_{ij} = 0, \quad s_j = 0 \quad (4)$$

In this case, $\mathbf{S} = 0$.

3 Sakaguchi Theorem

Here, we give a proof of Sakaguchi's Theorem. Our approach is completely different from Sakaguchi's method. For this aim, we remark the following.

Lemma 3.1. ([14]) The following Bianchi identities holds:

$$R^i_{jkl|m} + R^i_{jlm|k} + R^i_{jmk|l} = B^i_{jku} R^u_{lm} + B^i_{jlu} R^u_{km} + B^i_{klu} R^u_{jm}, \quad (5)$$

$$B^i_{jkl|m} - B^i_{jmk|l} = R^i_{jml,k}, \quad (6)$$

$$B^i_{jkl,m} = B^i_{jkm,l}, \quad (7)$$

where “ $|$ ” and “ $,$ ” denote the h - and v -covariant derivatives with respect to the Berwald connection, respectively.

Theorem 3.2. (Sakaguchi) For any Finsler metric of scalar flag curvature, there is a tensor D_{jkl} such that

$$D^i_{jkl|m} y^m = D_{jkl} y^i. \quad (8)$$

This means that every Weyl metric is a generalized Douglas-Weyl metric.

Proof. By definition, we have

$$D^i_{jkl} = B^i_{jkl} - \frac{2}{n+1} \{E_{jk} \delta^i_l + E_{kl} \delta^i_j + E_{lj} \delta^i_k + E_{jk,l} y^i\}. \quad (9)$$

Taking a horizontal derivation of (9) yields

$$D^i_{jkl|m} y^m = B^i_{jkl|m} y^m - \frac{2}{n+1} \{H_{jk} \delta^i_l + H_{kl} \delta^i_j + H_{lj} \delta^i_k + E_{jk,l|m} y^m y^i\}, \quad (10)$$

where $H_{ij} := E_{ij|m} y^m$ (see [11]). We have

$$R^i_{jkl} = \frac{1}{3} \left\{ \frac{\partial^2 R^i_k}{\partial y^j \partial y^l} - \frac{\partial^2 R^i_l}{\partial y^j \partial y^k} \right\}. \quad (11)$$

By assumption, F is of scalar curvature $\mathbf{K} = \mathbf{K}(x, y)$, which is equivalent to $R^i_k = \mathbf{K} F^2 h^i_k$. Plugging it into (11) gives

$$\begin{aligned} R^i_{jkl} &= \frac{1}{3} F^2 \{ \mathbf{K}_{,j,l} h^i_k - \mathbf{K}_{,j,k} h^i_l \} + \mathbf{K}_{,j} \{ F F_{y^l} h^i_k - F F_{y^k} h^i_l \} \\ &\quad + \frac{1}{3} \mathbf{K}_{,k} \{ 2 F F_{y^j} \delta^i_l - g_{jl} y^i - F F_{y^l} \delta^i_j \} + \mathbf{K} \{ g_{jl} \delta^i_k - g_{jk} \delta^i_l \} \\ &\quad + \frac{1}{3} \mathbf{K}_{,l} \{ 2 F F_{y^j} \delta^i_k - g_{jk} y^i - F F_{y^k} \delta^i_j \}. \end{aligned} \quad (12)$$

Differentiating (12) with respect to y^m gives a formula for $R^i_{jkl,m}$ expressed in terms of \mathbf{K} and its derivatives. Contracting (6) with y^k , we obtain

$$\begin{aligned} B^i_{jml|k}y^k &= 2\mathbf{K}C_{jlm}y^i - \frac{\mathbf{K}_{,j}}{3}\left\{FF_{y^l}\delta_m^i + FF_{y^m}\delta_l^i - 2g_{lm}y^i\right\} \\ &\quad - \frac{\mathbf{K}_{,l}}{3}\left\{FF_{y^j}\delta_m^i + FF_{y^m}\delta_j^i - 2g_{jm}y^i\right\} \\ &\quad - \frac{\mathbf{K}_{,m}}{3}\left\{FF_{y^j}\delta_l^i + FF_{y^l}\delta_j^i - 2g_{jl}y^i\right\} \\ &\quad - \frac{1}{3}F^2\left\{\mathbf{K}_{,j,m}h_l^i + \mathbf{K}_{,j,l}h_m^i + \mathbf{K}_{,l,m}h_j^i\right\}, \end{aligned} \quad (13)$$

where $C_{jlm} := \frac{1}{4}[F^2]_{y^jy^ly^m}$ denotes the Cartan torsion of F . For (13), see (11.24) in [14]. It follows from (13) that

$$H_{jk} = -\frac{n+1}{6}\left\{y_l\mathbf{K}_{,j} + y_j\mathbf{K}_{,l} + F^2\mathbf{K}_{,j,l}\right\}. \quad (14)$$

By plugging (13) and (14) in (10) we get

$$\begin{aligned} D^i_{jkl|m}y^m &= 2\mathbf{K}C_{jlm}y^i + \frac{2}{3}\left\{\mathbf{K}_{,j}g_{kl} + \mathbf{K}_{,l}g_{kj} + \mathbf{K}_{,k}g_{jl}\right\}y^i \\ &\quad + \frac{1}{3}\left\{y_{,j}\mathbf{K}_{,k,l} + y_{,k}\mathbf{K}_{,j,l} + y_{,l}\mathbf{K}_{,k,j}\right\}y^i - \frac{2}{n+1}E_{jk,l|m}y^my^i. \end{aligned} \quad (15)$$

Then every Weyl metric is a generalized Douglas-Weyl metric. \square

4 Proof of Theorem 1.1

In this section, we are going to prove Theorem 1.1. Indeed, we characterize regular generalized Douglas-Weyl metric (α, β) -metrics $F = \alpha\phi(s)$, $s = \beta/\alpha$, with vanishing S-curvature.

Proof of Theorem 1.1. Since F has vanishing S-curvature then by (4) we have $r_{ij} = 0$ and $s_j = 0$. Then (1) reduces to following

$$G^i = G_\alpha^i + \alpha Q s^i_0. \quad (16)$$

By differential (16) with respect to y^j , y^l and y^k we get

$$\begin{aligned} B^i_{jkl} &= s^i_l\left[Q\alpha_{jk} + Q_k\alpha_j + Q_j\alpha_k + \alpha Q_{jk}\right] + s^i_j\left[Q\alpha_{lk} + Q_k\alpha_l + Q_l\alpha_k + \alpha Q_{lk}\right] \\ &\quad + s^i_k\left[Q\alpha_{jl} + Q_j\alpha_l + Q_l\alpha_j + \alpha Q_{jl}\right] + s^i_0\left[\alpha_{jkl}Q + \alpha_{jk}Q_l + \alpha_{lk}Q_j + \alpha_{lj}Q_k\right] \\ &\quad + s^i_0\left[\alpha Q_{jkl} + \alpha_l Q_{jk} + \alpha_j Q_{lk} + \alpha_k Q_{jl}\right]. \end{aligned} \quad (17)$$

Contracting (17) with h^m_i implies that

$$\begin{aligned} h^m_i B^i_{jkl} &= s^m_0\left[\alpha_{jkl}Q + \alpha_{jk}Q_l + \alpha_{lk}Q_j + \alpha_{lj}Q_k + \alpha Q_{jkl} + \alpha_l Q_{jk} + \alpha_j Q_{lk} + \alpha_k Q_{jl}\right] \\ &\quad + \left(s^m_l - F^{-2}s^0_ly^m\right)\left[Q\alpha_{jk} + Q_k\alpha_j + Q_j\alpha_k + \alpha Q_{jk}\right] \\ &\quad + \left(s^m_j - F^{-2}s^0_jy^m\right)\left[Q\alpha_{lk} + Q_k\alpha_l + Q_l\alpha_k + \alpha Q_{lk}\right] \\ &\quad + \left(s^m_k - F^{-2}s^0_ky^m\right)\left[Q\alpha_{jl} + Q_j\alpha_l + Q_l\alpha_j + \alpha Q_{jl}\right]. \end{aligned} \quad (18)$$

Taking a horizontal derivation of Douglas curvature and contracting the result with h_i^m implies that

$$h_i^m D^i{}_{jkl|s} y^s = h_i^m B^i{}_{jkl|s} y^s - \frac{2}{n+1} \left\{ H_{jk} h_l^m + H_{kl} h_j^m + H_{lj} h_k^m \right\}. \quad (19)$$

By assumption $\mathbf{S} = 0$ and then $H_{ij} = 0$. Since $h_{i|s}^m = 0$, then (19) reduces to following

$$(h_i^m B^i{}_{jkl|s})_s y^s = h_i^m B^i{}_{jkl|s} y^s = 0. \quad (20)$$

By (18) and (20), we have

$$\begin{aligned} h_i^m B^i{}_{jkl|s} y^s = & s_{0|0}^m \left(\alpha_{jkl} Q + \alpha_{jk} Q_l + \alpha_{lk} Q_j + \alpha_{lj} Q_k + \alpha Q_{jkl} + \alpha_l Q_{jk} + \alpha_j Q_{lk} + \alpha_k Q_{jl} \right) \\ & + \left(s_l^m - F^{-2} s_l^0 y^m \right) \left(Q_{|0} \alpha_{jk} + Q_{k|0} \alpha_j + Q_{j|0} \alpha_k + \alpha Q_{jk|0} \right) \\ & + \left(s_j^m - F^{-2} s_j^0 y^m \right) \left(Q_{|0} \alpha_{lk} + Q_{k|0} \alpha_l + Q_{l|0} \alpha_k + \alpha Q_{lk|0} \right) \\ & + \left(s_k^m - F^{-2} s_k^0 y^m \right) \left(Q_{|0} \alpha_{jl} + Q_{j|0} \alpha_l + Q_{l|0} \alpha_j + \alpha Q_{jl|0} \right) \\ & + \left(s_{l|0}^m - F^{-2} s_{l|0}^0 y^m \right) \left(Q \alpha_{jk} + Q_k \alpha_j + Q_j \alpha_k + \alpha Q_{jk} \right) \\ & + \left(s_{j|0}^m - F^{-2} s_{j|0}^0 y^m \right) \left(Q \alpha_{lk} + Q_k \alpha_l + Q_l \alpha_k + \alpha Q_{lk} \right) \\ & + \left(s_{k|0}^m - F^{-2} s_{k|0}^0 y^m \right) \left(Q \alpha_{jl} + Q_j \alpha_l + Q_l \alpha_j + \alpha Q_{jl} \right) \\ & + s_0^m \left(\alpha_{jkl} Q_{|0} + \alpha_{jk} Q_{l|0} + \alpha_{lk} Q_{j|0} + \alpha_{lj} Q_{k|0} + \alpha Q_{jkl|0} \right) \\ & + s_0^m \left(\alpha_l Q_{jk|0} + \alpha_j Q_{lk|0} + \alpha_k Q_{jl|0} \right) = 0. \end{aligned} \quad (21)$$

Taking a horizontal derivation of $y_m s_0^m = 0$ implies that $y_{m|0} s_0^m + y_m s_{0|0}^m = 0$. Since $y_{m|0} = 0$, then we get $y_m s_{0|0}^m = 0$. Therefore by contracting (21) with y_m it follows that

$$\begin{aligned} (1 - \alpha^2 F^{-2}) \left[& s_l^0 \left(Q_{|0} \alpha_{jk} + Q_{k|0} \alpha_j + Q_{j|0} \alpha_k + \alpha Q_{jk|0} \right) \right. \\ & + s_j^0 \left(Q_{|0} \alpha_{lk} + Q_{k|0} \alpha_l + Q_{l|0} \alpha_k + \alpha Q_{lk|0} \right) \\ & + s_k^0 \left(Q_{|0} \alpha_{jl} + Q_{j|0} \alpha_l + Q_{l|0} \alpha_j + \alpha Q_{jl|0} \right) \\ & + s_{l|0}^0 \left(Q \alpha_{jk} + Q_k \alpha_j + Q_j \alpha_k + \alpha Q_{jk} \right) \\ & + s_{j|0}^0 \left(Q \alpha_{lk} + Q_k \alpha_l + Q_l \alpha_k + \alpha Q_{lk} \right) \\ & \left. + s_{k|0}^0 \left(Q \alpha_{jl} + Q_j \alpha_l + Q_l \alpha_j + \alpha Q_{jl} \right) \right] = 0. \end{aligned} \quad (22)$$

Since F is not Riemannian, then $(1 - F^{-2} \alpha^2) \neq 0$. Therefore (22) reduces to following

$$\begin{aligned} & s_l^0 \left(Q_{|0} \alpha_{jk} + Q_{k|0} \alpha_j + Q_{j|0} \alpha_k + \alpha Q_{jk|0} \right) + s_{l|0}^0 \left(Q \alpha_{jk} + Q_k \alpha_j + Q_j \alpha_k + \alpha Q_{jk} \right) \\ & + s_j^0 \left(Q_{|0} \alpha_{lk} + Q_{k|0} \alpha_l + Q_{l|0} \alpha_k + \alpha Q_{lk|0} \right) + s_{j|0}^0 \left(Q \alpha_{lk} + Q_k \alpha_l + Q_l \alpha_k + \alpha Q_{lk} \right) \\ & + s_k^0 \left(Q_{|0} \alpha_{jl} + Q_{j|0} \alpha_l + Q_{l|0} \alpha_j + \alpha Q_{jl|0} \right) + s_{k|0}^0 \left(Q \alpha_{jl} + Q_j \alpha_l + Q_l \alpha_j + \alpha Q_{jl} \right) = 0 \end{aligned} \quad (23)$$

Since $s_j = b_m s_j^m = 0$, then we have

$$0 = (s_j)_{|0} = (b_m s_j^m)_{|0} = b_{m|0} s_j^m + b_m s_{j|0}^m = (r_{m0} + s_{m0}) s_j^m + b_m s_{j|0}^m. \quad (24)$$

Since $r_{00} = 0$, then (24) reduces to following

$$b_m s_{j|0}^m = -s_{m0} s_j^m. \quad (25)$$

Multiplying (25) with y^j yields

$$b_m s_{0|0}^m = -s_{m0} s_0^m. \quad (26)$$

Contracting (25) with b^j implies that

$$b^j b_m s_{j|0}^m = 0. \quad (27)$$

By considering (26) and (27) and multiplying (23) with $b^j b^k b^l$ it follows that

$$s_0^m s_{m0} (\alpha_2 Q + 2\alpha_1 Q_1 + \alpha Q_2) = 0, \quad (28)$$

where

$$\alpha_1 := b^i \alpha_{y^i}, \quad \alpha_2 := b^i b^j \alpha_{y^i y^j}, \quad Q_1 := b^i Q_{y^i}, \quad Q_2 := b^i b^j Q_{y^i y^j}.$$

By (28), we have $Q\alpha_2 + 2\alpha_1 Q_1 + \alpha Q_2 = 0$ or $s_0^m s_{m0} = 0$. Suppose that, the first case holds

$$Q\alpha_2 + 2\alpha_1 Q_1 + \alpha Q_2 = 0. \quad (29)$$

By definition, we have the following

$$\alpha_{y^i} = \alpha^{-1} y_i, \quad \alpha_{y^j y^k} = \alpha^{-3} A_{jk}, \quad \alpha_{y^j y^k y^l} = -\alpha^{-5} [A_{jk} y_l + A_{jl} y_k + A_{lk} y_j],$$

where $A_{jk} := \alpha^2 a_{jk} - y_j y_k$. So we get

$$\alpha_1 = s, \quad (30)$$

$$\alpha_2 = (b^2 - s^2) \alpha^{-1}, \quad (31)$$

$$\alpha_3 = -3s(b^2 - s^2) \alpha^{-2}, \quad (32)$$

$$Q_1 = Q' \frac{(b^2 - s^2)}{\alpha}, \quad (33)$$

$$Q_2 = \frac{(b^2 - s^2) [(b^2 - s^2) Q'' - 3s Q']}{\alpha^2}, \quad (34)$$

$$Q_3 = \frac{3(b^2 - s^2) [(b^2 - s^2)^2 Q''' - 3s(b^2 - s^2) Q'' - (b^2 - 5s^2) Q']}{\alpha^3}. \quad (35)$$

By plugging (30), (31), (32), (33), (34) and (35) into (29), we get

$$(b^2 - s^2) Q'' + (Q - s Q') = 0,$$

which implies that

$$Q = ks + q\sqrt{b^2 - s^2}, \quad (36)$$

where k and q were constants. By (36), it follows that

$$\phi = c \exp \left[\int_0^s \frac{kt + q\sqrt{b^2 - t^2}}{1 + kt^2 + qt\sqrt{b^2 - t^2}} dt \right], \quad (37)$$

where c is a real constant. But, (37) is an almost regular (α, β) -metric (for more details, see [16]). Consequently, we conclude that $s_0^m s_{m0} = 0$ which implies that β is closed. Therefore by (17), F reduces to a Berwald metric. \square

Remark 4.1. In the Theorem 1.1, if we suppose that $F = \alpha\phi(s)$ is almost regular metric and $\mathbf{B} \neq 0$ then it reduces to (37), where $c > 0$, $q > 0$ and k are real constants. Since $\mathbf{S} = 0$, then F can not be a Douglas metric. By Cheng' Theorem (Theorem 4 in [4]), if F is a Weyl metric then it reduces to a Berwald metric. Thus F is not a Weyl metric. By Theorem 1.2 in [16], since $s_{ij} \neq 0$ then F is not a Landsberg metric, also.

Proof of Corollary 1.1: In [7], Cui shows that for the Matsumoto metric $F = \frac{\alpha^2}{\alpha-\beta}$ and the special (α, β) -metric $F = \alpha + \epsilon\beta + \kappa(\beta^2/\alpha)$ ($\kappa \neq 0$), $\mathbf{E} = \frac{n+1}{2}cF^{-1}\mathbf{h}$ if and only if $\mathbf{S} = 0$. Then by Theorem 1.1, we get the proof. \square

In [19], it is proved that every R-quadratic Finsler metric is a generalized Douglas-Weyl metric. Thus by Theorem 1.1, we get the following.

Corollary 4.1. *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be a regular (α, β) -metric on a manifold M with vanishing S -curvature. Suppose that F is not a Finsler metric of Randers type. Then F is R-quadratic if and only if it is a Berwald metric.*

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